CDI-II — Prática. Ficha 3. 15/3/21

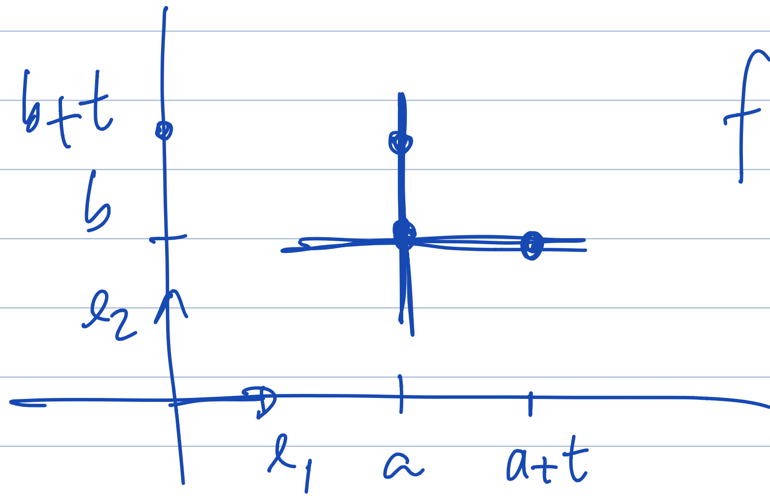
Matriz Jacobiana $\equiv Df(a)$

$\underset{m \times n}{Df(a)}$ $\qquad f: \mathbb{R}^n \to \mathbb{R}^m$

$$\begin{array}{cc} \begin{bmatrix} \vdots & - - - - & \vdots \\ - - - & \dfrac{\partial f_j}{\partial x_k}(a) & - \\ \vdots & - - - & \end{bmatrix}_{m \times n} & \leftarrow j = 1, \dots, m \\ \uparrow & \\ k = 1, \dots, n & \end{array}$$

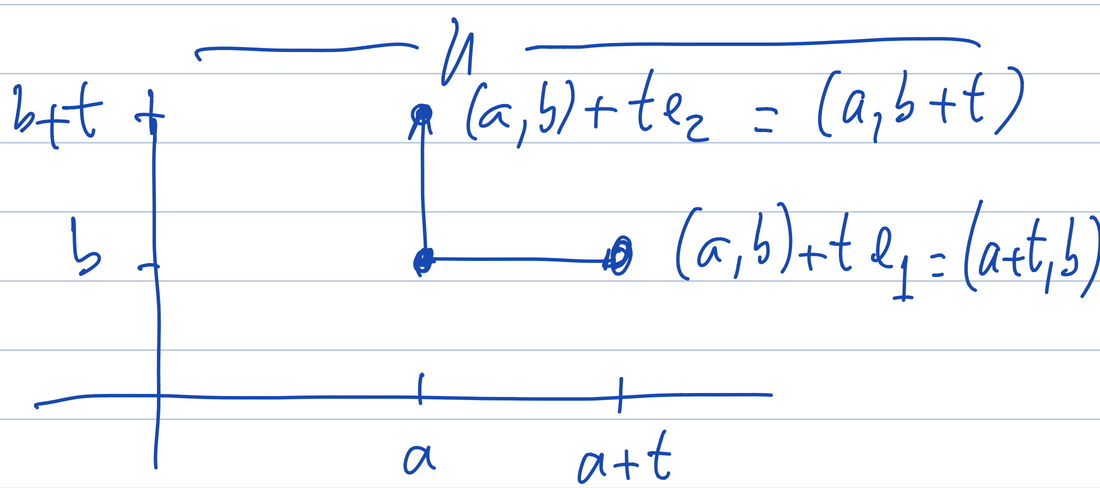
$$\frac{\partial f_j}{\partial x_k}(a) = \lim_{t \to 0} \frac{f_j(a + te_k) - f_j(a)}{t}$$

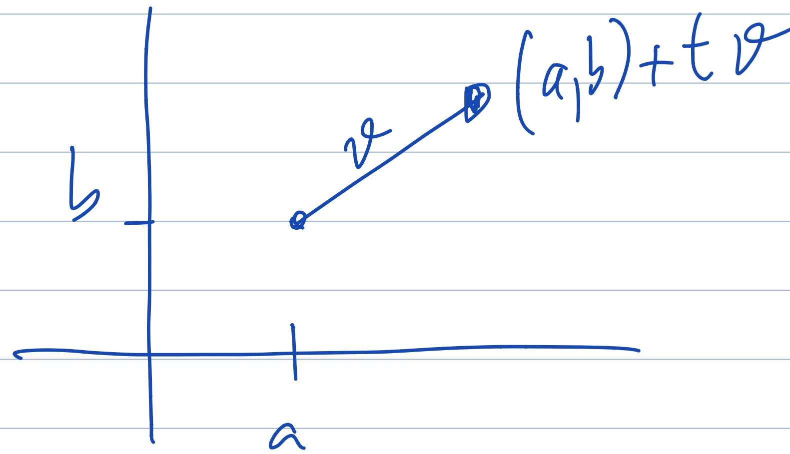
$f : \mathbb{R}^2 \to \mathbb{R}$



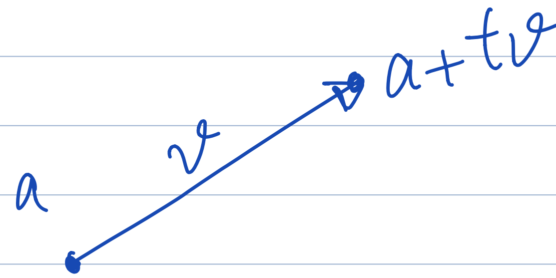
$$\frac{\partial f}{\partial x}(a,b) = \lim_{t \to 0} \frac{f(a+t,b) - f(a,b)}{t}$$

$$\frac{\partial f}{\partial y}(a,b) = \lim_{t \to 0} \frac{f(a,b+t) - f(a,b)}{t}$$





Em geral em $\mathbb{R}^n$.



$$\lim_{t \to 0} \frac{f(a+tv)-f(a)}{t}$$

$=$ derivada de f em a segundo o vector v

$$D_v f(a) \equiv \frac{\partial f}{\partial v}(a)$$

quando $v = e_k$

$$D_{e_k} f(a) \equiv \frac{\partial f}{\partial e_k}(a) \equiv \frac{\partial f}{\partial x_k}(a)$$

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$f: \mathbb{R}^n \to \mathbb{R}^m$, diferenciável em $a \in \mathbb{R}^n$ se $\exists\, Df(a): \mathbb{R}^n \to \mathbb{R}^m$ linear t.q.

$$f(a+h) - f(a) - Df(a)h = o(h)$$

$\boxed{\text{Se } f \text{ for dif.}}$ em a, então

$h = tv$, v dado.

$$f(a+tv) - f(a) - Df(a)(tv) = o(tv)$$

$$f(a+tv) - f(a) - t\,Df(a)v = o(tv)$$

$$\frac{f(a+tv) - f(a)}{t} = \underbrace{Df(a)v} + \frac{o(tv)}{t}$$

$t \to 0$

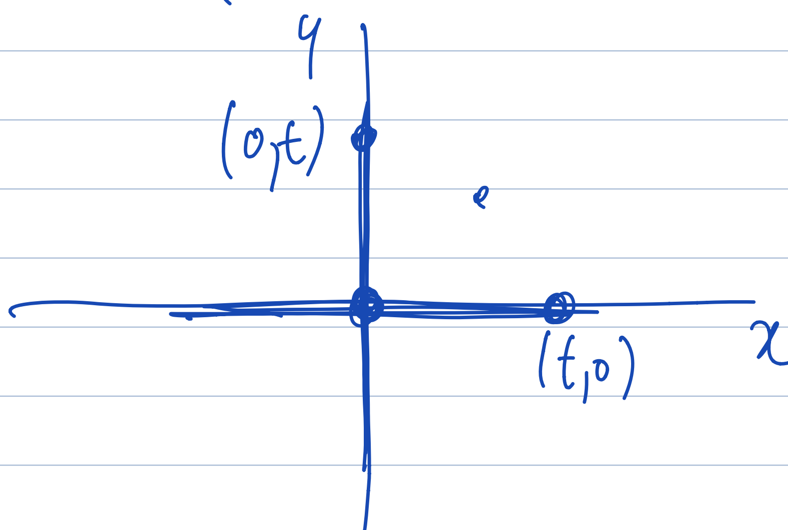
$$\boxed{\frac{\partial f}{\partial v}(a) = Df(a)v} + 0$$

Ficha 3:

1) trivial

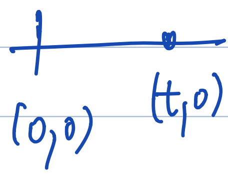
2) $\dfrac{\partial f}{\partial x}(0,0)$, $\dfrac{\partial f}{\partial y}(0,0)$

$$f(x,y) = \begin{cases} \dfrac{y^3}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$



Usar a definição de derivada parcial

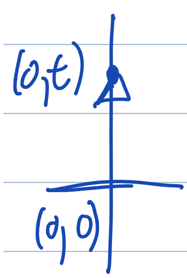
$$\frac{\partial f}{\partial x}(0,0) = \lim_{t \to 0} \frac{f(t,0) - f(0,0)}{t}$$



$$= \lim_{\boxed{t \to 0}} \boxed{\frac{0 - 0}{t}} = 0.$$

$$(0,0) + t e_x = (0,0) + (t,0) = (t,0)$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t \to 0} \frac{f(0,t) - f(0,0)}{t}$$



$$= \lim_{t \to 0} \frac{\frac{t^3}{t^2}}{t} = 1 \quad //$$

$$(0,0) + t e_y = (0,0) + (0,t) = (0,t)$$

3 – simples ✓

4 – $\frac{\partial f}{\partial v}(a) = Df(a)\,v$

se f for diferenciável em a.

a) $f(x,y) = y^x$, $y>0$, $y = e^{\ln y}$

$f(x,y) = e^{x\ln(y)}$

$\frac{\partial f}{\partial x}(2,1) = 0$ $\frac{\partial f}{\partial y}(2,1) = 2$

$\frac{\partial f}{\partial x}(x,y) = \ln y\, e^{x\ln y}$; $\frac{\partial f}{\partial y}(x,y) = \frac{x}{y}\, e^{x\ln y}$

$$\frac{d}{dx} f(u(x)) = f'(u(x))\, u'(x).$$

$$\text{————————— // —————————}$$

$$Df(2,1) = \begin{bmatrix} 0 & 2 \end{bmatrix} \quad ; \; v = (1,1)$$

$$\frac{\partial f}{\partial v}(2,1) = \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= 2 \; //.$$

$$\text{————————— // —————————}$$

4 - b) $g(x,y,z) = e^z + xy$

$$Dg(x,y,z) = \begin{bmatrix} y & x & e^z \end{bmatrix}$$

$$Dg(1,1,1) = \begin{bmatrix} 1 & 1 & e \end{bmatrix}$$

$$v = (1,-1,1)$$

$$\frac{\partial g}{\partial v}(1,1,1) = Dg(1,1,1)\,v$$

$$= \begin{bmatrix} 1 & 1 & e \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = e$$

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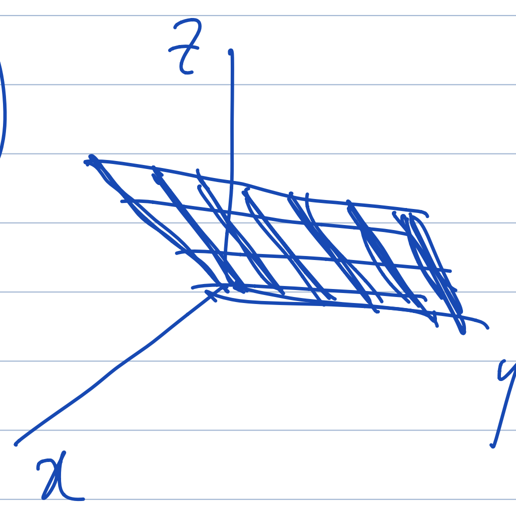
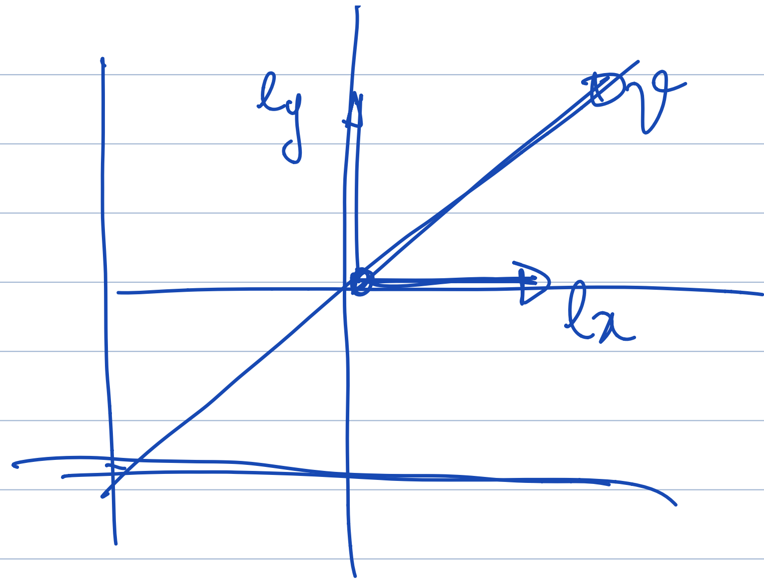
$$f(x,y) = x$$

$$\boxed{z = x}$$





5- $v = ?$ $\quad \dfrac{\partial f}{\partial v}(1,2) = 0$

$$f(x,y) = x(y^2 + xy) = xy^2 + x^2 y$$

$$\underbrace{\frac{\partial f}{\partial v}(1,2)}_{=\,0} = Df(1,2)\, v$$

?

$$f(x,y) = x^2 y + y^2 x$$

$$\frac{\partial f}{\partial x}(x,y) = 2xy + y^2$$

$$\frac{\partial f}{\partial y}(x,y) = x^2 + 2xy$$

$$Df(1,2) = \begin{bmatrix} 8 & 5 \end{bmatrix}$$

$$0 = \begin{bmatrix} 8 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 8v_1 + 5v_2$$

$$\boxed{v = (5, -8)} \text{ etc.}$$

6 - Note: $\sqrt{\cdot}$ não tem derivada na origem.

$$f(x,y) = y\sqrt{x^2+y^2}$$

Usar a definição:

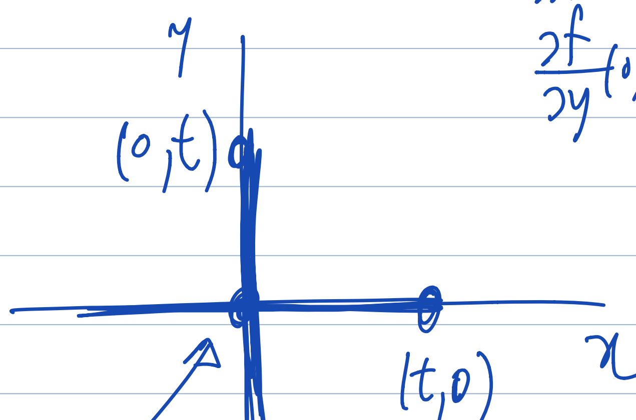
$$f(a+h) - f(a) - \underbrace{Df(a)}_{\text{matriz das derivadas parciais}}\,h = o(h)$$

$\frac{\partial f}{\partial x}(0,0)$

$\frac{\partial f}{\partial y}(0,0)$

$a = (0,0)$

$h = (x,y)$



$$f(x,y) = y\sqrt{x^2+y^2}$$

$f(0,0) = 0$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{0}{t} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{t\sqrt{t^2}}{t}$$

$$= \lim_{t\to 0} |t| = 0.$$

$$Df(0,0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$f(x,y) - \underbrace{f(0,0)}_{=0} - \underbrace{Df(0,0)(x,y)}_{\begin{bmatrix}0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 0} = o(x,y)$$

$$f(x,y) \overset{?}{=} o(x,y)$$

i.e. $$\lim_{(x,y)\to(0,0)} \frac{|f(x,y)|}{\|(x,y)\|} \overset{?}{=} 0$$

$$\frac{|f(x,y)|}{\|(x,y)\|} = \frac{|y\sqrt{x^2+y^2}|}{\|(x,y)\|}$$

$$= |y| \longrightarrow 0$$

$\therefore$ f é diferenciável em $(0,0)$

e $Df(0,0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$.

~~Se f for dif. em a~~ !! , então

$$f(a+h) - f(a) - Df(a)h = o(h)$$

$$\underbrace{f(a+h) - f(a)} = \underbrace{Df(a)h}_{\substack{\downarrow\, h\to 0 \\ 0}} + \underbrace{o(h)}_{\substack{\downarrow \\ 0}}$$

$$o(h) = \underbrace{\frac{o(h)}{\|h\|}}_{\to 0} \|h\| \longrightarrow 0$$

$x = a + h$

$$f(x) \xrightarrow[\text{qd } x \to a]{} f(a)$$

f é contínua.

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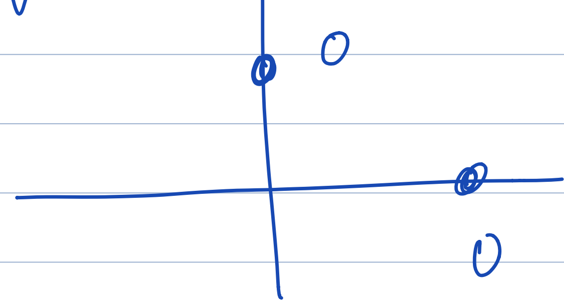
7-i) f não é contínua em $(0,0)$

$\Longrightarrow$ f não é dif. em $(0,0)$

7-ii) g é contínua.



$$\underbrace{g(x,y) - g(0,0)}_{=0} - \underbrace{Dg(0,0)(x,y)}_{=0} = \theta(x,y)$$

$$\frac{|g(x,y)|}{\|(x,y)\|} = \frac{|xy|}{x^2+y^2}$$

não tem limite.

g não é diferenciável em $(0,0)$

8-c) $$\frac{\partial f}{\partial v}(0,0) = \lim_{t \to 0} \frac{f(2t,3t) - f(0,0)}{t}$$

$v = (2,3)$ $\qquad$ $a = (0,0)$

$tv = (2t,3t)$ $\qquad$ $a + tv = (2t,3t)$